San José State University

Math 263: Stochastic Processes

Branching processes

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This lecture is based on the following textbook sections:

- Section 4.7

Outline of the presentation

- What is a branching process?
- Expectation and variance of X_n
- Probability of the population dying out

Consider a population consisting of individuals able to produce offspring of the same kind.

Suppose that each individual will, by the end of its lifetime, have produced j new offspring with probability p_j independently of the numbers produced by other individuals:

0	1	2	$\ldots$	j	$\ldots$
p_0	p_1	p_2	$\ldots$	p_j	$\ldots$

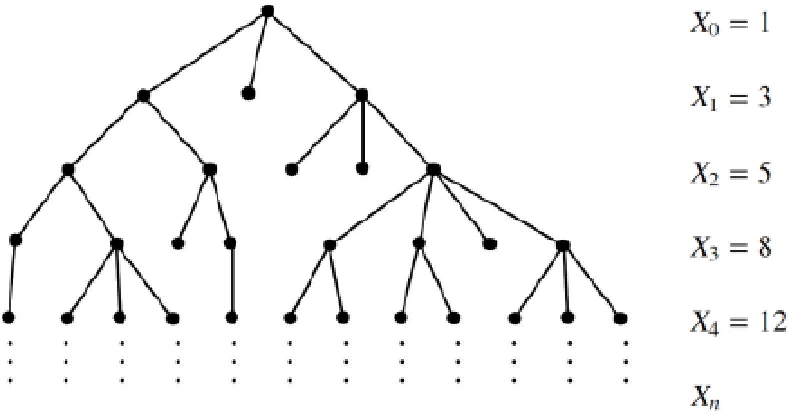
In the above, $0 \le p_j < 1$ for all j and $\sum_{j\ge 0} p_j = 1$

The number of individuals initially present, denoted by X_0, is called the size of the zeroth generation.

All offspring of the zeroth generation constitute the first generation and their number is denoted by X_1.

In general, let X_n denote the size of the nth generation.

It follows that $\{X_n, n = 0, 1, \ldots\}$ is a Markov chain with state space $S = Z_0^+$, the set of nonnegative integers.


$X_0 = 1$
$X_1 = 3$
$X_2 = 5$
$X_3 = 8$
$X_4 = 12$
X_n

Let's sketch the chain below:

This Markov chain is not irreducible:

- State 0 is recurrent, since it is absorbing.
- If $p_0 > 0$, then $p_{i0} = p_0^i > 0$ for all $i > 0$, implying that all other states are transient.

Since any finite set of transient states $\{1, 2, \ldots, n\}$ will be visited only finitely often, this leads to the important conclusion that, if $p_0 > 0$, then the population will either die out or its size will converge to infinity.

Let $Y_{n,i}$ be the number of offspring of a single individual i in the nth generation: $P(Y_{n,i} = j) = p_j$ for all $j \geq 0$.

Then for any fixed $n \geq 1$,

$$X_{n+1} = \sum_{i=1}^{X_n} Y_{n,i}$$

and $\{Y_{n,i}\}_i$ are iid with the same expected value and variance

$$\mu = \sum_{j=0}^{\infty} j p_j, \quad \sigma^2 = \sum_{j=0}^{\infty} (j-\mu)^2 p_j$$

Theorem 0.1. Suppose that $X_0 = 1$. Then, for any $n \geq 1$,

$$E(X_n) = \mu^n, \quad \mathrm{Var}(X_n) = \begin{cases} n\sigma^2, & \mu = 1 \\ \frac{\mu^{n-1}(1-\mu^n)}{1-\mu}\sigma^2, & \mu \geq 1 \end{cases}$$

Proof. We condition on X_{n-1} to obtain

$$E(X_n) = E(E(X_n \mid X_{n-1})) = E(\mu X_{n-1}) = \mu E(X_{n-1}), \quad n \geq 1.$$

Combining with $E(X_0) = 1$ we obtain that $E(X_n) = \mu^n$ for all $n \geq 1$.

Similarly, we condition on X_{n-1} to calculate the variance of X_n:

$$\begin{aligned}\mathrm{Var}(X_n) &= \mathrm{E}(\mathrm{Var}(X_n \mid X_{n-1})) + \mathrm{Var}(\mathrm{E}(X_n \mid X_{n-1})) \\ &= \mathrm{E}(\sigma^2 X_{n-1}) + \mathrm{Var}(\mu X_{n-1}) \\ &= \sigma^2 \mu^{n-1} + \mu^2 \mathrm{Var}(X_{n-1})\end{aligned}$$

By applying the formula recursively with $n = 1, 2, \ldots$, we obtain that

$$\mathrm{Var}(X_n) = \sigma^2(\mu^{n-1} + \mu^n + \cdots + \mu^{2n-2}) = \begin{cases} n\sigma^2, & \mu = 1 \\ \frac{\mu^{n-1}(1-\mu^n)}{1-\mu}\sigma^2, & \mu \geq 1 \end{cases}$$

Lastly, we study the probability that the population will eventually die out (under the assumption that $X_0 = 1$):

$$\pi_0 = \lim_{n\to\infty} P(X_n = 0 \mid X_0 = 1) = \lim_{n\to\infty} p_{10}^{(n)}.$$

Theorem 0.2. If $\mu \le 1$, then $\pi_0 = 1$; otherwise (i.e., $\mu > 1$), $\pi_0 < 1$ is the smallest positive root of

$$x = \sum_{j=0}^{\infty} p_j x^j$$

<u>Proof</u> We condition on the number of offspring of the initial individual:

$$
\begin{aligned}
\pi_0 &= P(\text{population dies out} \mid X_0 = 1) \\
&= \sum_{j\geq 0} P(\text{population dies out} \mid X_1 = j, \cancel{X_0 = 1}) P(X_1 = j \mid X_0 = 1) \\
&= \sum_{j\geq 0} \pi_0^j \cdot p_j
\end{aligned}
$$

This shows that π_0 is a root of the following equation:

$$
x = \sum_{j=0}^{\infty} p_j x^j
$$

Let

$$g(x) = x - \sum_{j=0}^{\infty} p_j x^j, \quad 0 \le x \le 1.$$

Then

$$g(0) = -p_0 < 0, \quad g(1) = 0$$

Additionally,

$$g'(x) = 1 - \sum_{j=1}^{\infty} j p_j x^{j-1}, \quad g''(x) = -\sum_{j=2}^{\infty} j(j-1) p_j x^{j-2} < 0$$

and in particular,

$$g'(0) = 1 - p_1 > 0, \quad g'(1) = 1 - \mu.$$

If $\mu \leq 1$, then

$$g'(1) \geq 0, \qquad g'(x) > g'(1) \geq 0 \quad \text{for all } 0 < x < 1.$$

Thus, $x = 1$ must be the only zero of $g(x)$ on $[0,1]$, implying that $\pi_0 = 1$.

On the other hand, if $\mu > 1$, then $g'(1) = 1 - \mu < 0$.

It follows that there exists some $0 < c < 1$ such that $g'(c) = 0$ and thus $g(c)$ is the absolute maximum of $g(x)$ on $[0,1]$. In particular, $g(c) > g(1) = 0$.

Since $g(0) < 0$, by continuity, there exist a number $0 < r < c < 1$ such that $g(r) = 0$. This indicates that $g(x) = 0$ has a root $r \in (0,1)$

We conclude that $\pi_0 = r < 1$ because $\mathrm{E}(X_n) = \mu^n \to \infty$ (which requires that $\pi_0 < 1$).

Remark. It can also be shown that π_0 must be the smallest positive number satisfying

$$x = \sum_{j=0}^{\infty} p_j x^j$$

See Chapter 4 Problem 65 (page 287).

Example 0.1. Determine π_0 in each case below:

- $p_0 = \frac{1}{3}, p_1 = \frac{1}{2}, p_2 = 0, p_3 = \frac{1}{6}$ (answer: $\pi_0 = 1$)

- $p_0 = \frac{1}{4}, p_1 = \frac{1}{4}, p_2 = \frac{1}{2}$ (answer: $\pi_0 = \frac{1}{2}$)

Example 0.2. For each branching process in the preceding example, what is the probability that the population will die out if it initially consists of n individuals?